

# ADAPTATIVE DECOMPOSITION: THE CASE OF THE DRURY-ARVESON SPACE

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**ABSTRACT.** The maximum selection principle allows to give expansions, in an adaptive way, of functions in the Hardy space  $\mathbf{H}_2$  of the disk in terms of Blaschke products. The expansion is specific to the given function. Blaschke factors and products have counterparts in the unit ball of  $\mathbb{C}^N$ , and this fact allows us to extend in the present paper the maximum selection principle to the case of functions in the Drury-Arveson space of functions analytic in the unit ball of  $\mathbb{C}^N$ . This will give rise to an algorithm which is a variation in this higher dimensional case of the greedy algorithm. We also introduce infinite Blaschke products in this setting and study their convergence.

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## 1. INTRODUCTION

In [15] the authors introduced an algorithm based on the maximum selection principle, to decompose a given function of the Hardy space  $\mathbf{H}_2(\mathbb{D})$  of the unit disk into intrinsic components which correspond to modified Blaschke products

$$(1.1) \quad B_n(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - z\overline{a_n}} \prod_{k=1}^{n-1} \frac{z - a_k}{1 - z\overline{a_k}}, \quad n = 1, 2, \dots$$

where the points  $a_n \in \mathbb{D}$  are adaptively chosen according to the given function. These points  $a_n$  do not necessarily satisfy the so-called hyperbolic non-separability condition

$$(1.2) \quad \sum_{n=1}^{\infty} 1 - |a_n| = \infty,$$

and so the functions  $B_n(z)$  do not necessarily form a complete system in  $\mathbf{H}_2(\mathbb{D})$ . This decomposition may be obtained in an adaptive way, see [14], making the algorithm more

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efficient than the greedy algorithm of which it is a variation.

In [4] the above algorithm is extended to the matrix-valued case and the choice of a point and of a projection is based at each step on the maximal selection principle. The extension is possible because of the existence of matrix-valued Blaschke factors and is based on the existence of solutions of interpolation problems in the matrix-valued Hardy space of the disk.

When leaving the realm of one complex variable, a number of possibilities occur, and in particular the unit ball  $\mathbb{B}_N$  of  $\mathbb{C}^N$  and the polydisk. The polydisk case will be studied in a future publication. In this paper we focus on the case of the unit ball. For the present purposes, it is more convenient to consider the Drury-Arveson space rather than the Hardy space of the ball, and we extend some of the results of [15] and [4] to the setting of the Drury-Arveson space, denoted here  $\mathbf{H}(\mathbb{B}_N)$ . This is the space with reproducing kernel

$$\frac{1}{1 - \langle z, w \rangle}, \quad z, w \in \mathbb{B}_N,$$

with

$$\langle z, w \rangle = \sum_{u=1}^N z_u \overline{w_u} = zw^*,$$

where  $z = (z_1, \dots, z_N)$  and  $w = (w_1, \dots, w_N)$  belong to  $\mathbb{B}_N$ . This space has a long history (see for instance [8, 1, 2, 10, 12, 17]) and is used in the proof of a von Neumann inequality for row contractions. Interpolation inside the space  $\mathbf{H}(\mathbb{B}_N)$  was done in [7]. A key tool in [7] was the existence in the ball of the counterpart of a Blaschke factor (appearing in [16]; see (2.5) below). The existence of these Blaschke factors and the fact that one can solve interpolation problems in  $\mathbf{H}(\mathbb{B}_N)$  allow us to develop the asserted extension.

The approach in [7] is based on the solution of Gleason's problem. For completeness we recall that given a space, say  $\mathcal{F}$ , of functions analytic in  $\Omega \subset \mathbb{C}^N$ , Gleason's problem consists in finding for every  $f \in \mathcal{F}$  and  $a = (a_1, \dots, a_N) \in \Omega$ , functions  $g_1(z, a), \dots, g_N(z, a) \in \mathcal{F}$  and such that

$$(1.3) \quad f(z) - f(a) = \sum_{u=1}^N (z_u - a_u) g_u(z, a), \quad z \in \Omega.$$

Using power series, one sees that there always exist analytic functions satisfying (1.3). The requirement is that one can choose them in  $\mathcal{F}$ .

The paper consists of five sections besides the introduction. In Sections 2 and 3 we review some basic facts on the Drury-Arveson space, and on the interpolation in it. The latter will be necessary to prove the maximum selection principle. This principle is proved in Section 4. In Section 5 we prove the convergence of the algorithm. In the last section, which is of independent interest, we consider infinite Blaschke products. When  $N > 1$  the  $a_n$  in (1.2) are vectors in  $\mathbb{B}_N$  and condition (1.2) is replaced by the requirement

$$\sum_{n=1}^{\infty} \sqrt{1 - a_n a_n^*} < \infty.$$

We note that most of the analysis presented here still holds for general complete Nevanlinna-Pick kernels, that is kernels of the form

$$\frac{1}{c(z)\overline{c(w)} - \langle d(z), d(w) \rangle_{\mathcal{H}}},$$

where  $c$  is scalar and  $d$  is  $\mathcal{H}$ -valued where  $\mathcal{H}$  is some Hilbert space or more generally, in some reproducing kernel Hilbert spaces in which Gleason's problem is solvable with bounded operators; see [5] for the latter.

## 2. THE DRURY-ARVESON SPACE

We use the multi-index notations

$$z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}, \quad \text{and} \quad \alpha! = \alpha_1! \cdots \alpha_N!,$$

with  $z = (z_1, \dots, z_N) \in \mathbb{C}^N$  and  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ . For  $z, w \in \mathbb{B}_N$  we have

$$(2.1) \quad \frac{1}{1 - zw^*} = \sum_{\alpha \in \mathbb{N}_0^N} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha.$$

The function (2.1) is thus positive definite in  $\mathbb{B}_N$ . The associated reproducing kernel Hilbert space, which we denote by  $\mathbf{H}(\mathbb{B}_N)$ , is called the Drury-Arveson space, and can be characterized as

$$(2.2) \quad \mathbf{H}(\mathbb{B}_N) = \left\{ f(z) = \sum_{\alpha \in \mathbb{N}_0^N} z^\alpha f_\alpha : \|f\|_{\mathbf{H}(\mathbb{B}_N)}^2 = \sum_{\alpha \in \mathbb{N}_0^N} \frac{\alpha!}{|\alpha|!} |f_\alpha|^2 < \infty \right\}.$$

For  $N > 1$  the Drury-Arveson space is contractively included in, but different from, the Hardy space of the ball. The latter has reproducing kernel

$$\frac{1}{(1 - \langle z, w \rangle)^N}, \quad z, w \in \mathbb{B}_N.$$

See [6] for an expression for the inner product (not in terms of a surface integral).

We define  $(\mathbf{H}(\mathbb{B}_N))^{n \times m}$  as in (2.2), but with now  $f_\alpha, g_\alpha \in \mathbb{C}^{n \times m}$  and define for  $f, g \in (\mathbf{H}(\mathbb{B}_N))^{n \times m}$ , with  $g(z) = \sum_{\alpha \in \mathbb{N}_0^N} z^\alpha g_\alpha$ ,

$$(2.3) \quad [f, g]_{(\mathbf{H}(\mathbb{B}_N))^{n \times m}} = \sum_{\alpha \in \mathbb{N}_0^N} g_\alpha^* f_\alpha, \quad \text{and}$$

$$(2.4) \quad \langle f, g \rangle_{(\mathbf{H}(\mathbb{B}_N))^{n \times m}} = \text{Tr} [f, g]_{(\mathbf{H}(\mathbb{B}_N))^{n \times m}}.$$

In the sequel we will not write anymore explicitly the space in these forms. We will write sometimes  $\mathbb{C}^N$  instead of  $\mathbb{C}^{1 \times N}$ .

For  $a \in \mathbb{B}_N$  we will use the notations  $e_a$  and  $b_a$  for the normalized Cauchy kernel and the  $\mathbb{C}^N$ -valued Blaschke factor at the point  $a$  respectively, that is:

$$(2.5) \quad e_a(z) = \frac{\sqrt{1 - \|a\|^2}}{1 - \langle z, a \rangle} \quad \text{and} \quad b_a(z) = \frac{(1 - \|a\|^2)^{1/2}}{1 - \langle z, a \rangle} (z - a)(I_N - a^* a)^{-1/2}.$$

Let  $w \in \mathbb{B}_N$ . Then (see [16]; another more analytic and maybe easier proof can be found in [7]):

$$(2.6) \quad \frac{1 - b_a(z)b_a(w)^*}{1 - zw^*} = \frac{1 - aa^*}{(1 - za^*)(1 - w^*a)}, \quad z, w \in \mathbb{B}_N.$$

Gleason's problem is solvable in the Drury-Arveson space and in the Hardy space; see [5]. For  $a = 0$  and by setting  $g_u(z, 0) = g_u(z)$ , a solution is given by

$$g_u(z, 0) = \int_0^1 \frac{\partial}{\partial z_u} f(tz) dt = \sum_{\alpha \in \mathbb{N}_0^N} \frac{\alpha_u}{|\alpha|} z^{\alpha - \epsilon_u},$$

where  $\epsilon_u$  is the  $N$ -index with all the other entries equal to 0, but the  $u$ -th one equal to 1, and with the understanding that

$$\frac{\alpha_u}{|\alpha|} z^{\alpha - \epsilon_u} = 0$$

if  $\alpha_u = 0$ . We set  $(R_u f)(z) = \int_0^1 \frac{\partial}{\partial z_u} f(tz) dt$ . We thus have

$$f(z) - f(0) = \sum_{u=1}^N z_u (R_u f)(z).$$

When  $N = 1$ , then  $R_1$  reduces to the classical backward-shift operator which to  $f$  associates the function  $\frac{f(z)-f(0)}{z}$  for  $z \neq 0$  and  $f'(0)$  for  $z = 0$ .

### 3. INTERPOLATION IN THE DRURY-ARVESON SPACE

This section is based on [7] and reviews the tools necessary to develop the maximum selection principle and the convergence result in the next section. We provide the proofs for completeness.

**Proposition 3.1.** *Let  $0 \neq c \in \mathbb{C}^{n \times 1}$ ,  $a \in \mathbb{B}_N$ , and let  $f \in \mathbf{H}(\mathbb{B}_N)^{n \times 1}$ . Then*

$$c^* f(a) = 0 \iff f(z) = B(z)g(z),$$

where  $B$  is given by

$$(3.1) \quad B(z) = U \begin{pmatrix} b_a(z) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix},$$

where  $b_a(z) \in \mathbb{C}^{1 \times N}$ ,  $U \in \mathbb{C}^{n \times n}$  is a unitary matrix with the first column equal to  $\frac{c}{c^*c}$ , and  $g$  is an arbitrary element of  $\mathbf{H}(\mathbb{B}_N)^{(N+n-1) \times 1}$ .

*Proof.* We recall the proof of the proposition; see [7, Proposition 4.5, p. 15]. We note that

$$c^* U = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and hence

$$c^* B(a) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0_{1 \times N} & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix} = 0_{n \times (N+(n-1))}.$$

and so every function of the form  $Bg$  with  $g \in \mathbf{H}(\mathbb{B}_N)^{(N+n-1) \times 1}$  is a solution of the interpolation problem. To prove the converse statement, we first remark that

$$\frac{I_n - B(z)B(w)^*}{1 - zw^*} = \frac{cc^*}{c^*c} \frac{1 - aa^*}{(1 - za^*)(1 - w^*a)}.$$

It follows that the one dimensional subspace  $\mathcal{H}_1$  of  $\mathbf{H}(\mathbb{B}_N)^n$  spanned by the vector  $\frac{\frac{c}{c^*c}}{1 - za^*}$  has reproducing kernel  $\frac{I_n - B(z)B(w)^*}{1 - zw^*}$ . Thus the decomposition of kernels

$$\frac{I_n}{1 - zw^*} = \frac{I_n - B(z)B(w)^*}{1 - zw^*} + \frac{B(z)B(w)^*}{1 - zw^*}$$

leads to an orthogonal decomposition of the space  $\mathbf{H}(\mathbb{B}_N)^n$  as

$$\mathbf{H}(\mathbb{B}_N)^n = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp,$$

where  $\mathcal{H}_1^\perp$  is the subspace of  $\mathbf{H}(\mathbb{B}_N)^n$  consisting of functions  $g$  such that  $c^*g(a) = 0$ . Since the reproducing kernel of  $\mathcal{H}_1^\perp$  is  $\frac{B(z)B(w)^*}{1 - zw^*}$  we have

$$\mathcal{H}_1^\perp = \{Bg; g \in \mathbf{H}(\mathbb{B}_N)^{(N+n-1)}\},$$

with norm

$$\|Bg\|_{\mathbf{H}(\mathbb{B}_N)^n} = \inf_{g \in \mathbf{H}(\mathbb{B}_N)^{(N+n-1)}} \|g\|_{\mathbf{H}(\mathbb{B}_N)^{(N+n-1)}}.$$

□

We note that we do not write the dependence of  $B$  on  $a$  and  $c$ .

**Definition 3.2.** *The  $\mathbb{C}^{n \times (N+n-1)}$ -valued function  $B$  is an elementary Blaschke factor. A (possibly infinite) Blaschke product is a product of terms of the form (3.1) of compatible (growing) sizes.*

**Remark 3.3.** Let  $\mathcal{B}$  be a  $\mathbb{C}^{n \times m}$ -valued Blaschke product (or taking values operators from  $\mathbb{C}^n$  into  $\ell_2$  if  $m = \infty$ ). Then  $\mathcal{B}$  is a Schur multiplier, meaning that the kernel  $\frac{I_n - \mathcal{B}(z)\mathcal{B}(w)^*}{1 - \langle z, w \rangle}$  is positive definite in  $\mathbb{B}_N$ . When  $N > 1$ , the family of Schur multipliers is strictly included in the family of functions analytic and contractive in the unit ball. For the realization theory of Schur multipliers, see for instance [9, 10].

More generally than (3.1) we have (see [7, Theorem 5.2, p. 17]):

**Theorem 3.4.** *Given  $a_1, \dots, a_M \in \mathbb{B}_N$  and vectors  $c_1, \dots, c_M \in \mathbb{C}^{n \times 1}$  different from  $0_{n \times 1}$ , a function  $f \in \mathbf{H}(\mathbb{B}_N)^{n \times 1}$  satisfies*

$$c_j^* f(a_j) = 0, \quad j = 1, \dots, M$$

*if and only if it is of the form  $f(z) = B(z)u(z)$ , where  $B(z)$  is a rational  $\mathbb{C}^{n \times (n+k(N-1))}$ -valued function, for some integer  $k \leq M$ , taking coisometric values on the boundary of  $\mathbb{B}_N$ , and  $u$  is an arbitrary element in  $\mathbf{H}(\mathbb{B}_N)^{(n+k(N-1)) \times 1}$ .*

*Proof.* Indeed, starting with  $j = 1$  we have that  $f = B_1 g_1$ , where  $B_1$  is given by (3.1) with  $a = a_1$  (and an appropriately constructed matrix  $U$ ) and  $g_1 \in \mathbf{H}(\mathbb{B}_N)^{(N+n-1) \times 1}$ . The interpolation condition  $c_2^* f(a_2) = 0$  becomes

$$(3.2) \quad c_2^* B_1(a_2) g_1(a_2) = 0.$$

If  $c_2^* B_1(a_2) = 0_{1 \times (N+n-1)}$ , any  $g_1$  will be a solution. Otherwise, we solve (3.2) using Proposition 3.1 and get

$$g_1(z) = B_2(z)g_2(z),$$

where  $B_2$  is  $\mathbb{C}^{(n+(N-1)) \times (n+2(N-1))}$ -valued and obtained from (3.1) with  $a = a_2$  and an appropriately constructed matrix  $U$ . Iterating this procedure we obtain the result. The fact that  $k$  may be strictly smaller than  $M$  comes from the possibility that conditions as (3.2) occur. This will not happen when  $N = 1$  and when all the  $a_j$  chosen are different.  $\square$

#### 4. THE MAXIMUM SELECTION PRINCIPLE

The proof is similar to the one in the original paper [15] and in [4], but one relevant difference is the use of orthogonal projections in  $\mathbb{C}^{n \times n}$  of fixed rank. The fact that the set of such projections is compact in  $\mathbb{C}^{n \times n}$  ensures the existence of a maximum. Besides the use of the normalized Cauchy kernel, the possibility of approximating by polynomials is a key tool in the proof.

**Proposition 4.1.** *Let  $B$  be a  $\mathbb{C}^{u \times n}$ -valued rational function of the variables  $z_1, \dots, z_N$ , analytic in an neighborhood of the closed unit ball  $\mathbb{B}_N$ , and taking co-isometric values on the unit sphere, let  $r_0 \in \{1, \dots, n\}$ , and let  $F \in \mathbf{H}(\mathbb{B}_N)^{n \times m}$ . There exists  $w_0 \in \mathbb{B}_N$  and a  $\mathbb{C}^{n \times n}$ -valued orthogonal projection  $P_0$  of rank  $r_0$  such that*

$$(1 - \|w_0\|^2) (\text{Tr} [B(w_0)P_0F(w_0), B(w_0)P_0F(w_0)]) \text{ is maximum.}$$

*Proof.* We first recall that for  $f \in \mathbf{H}(\mathbb{B}_N)$  (that is,  $n = m = 1$ ), with power series  $f(z) = \sum_{\alpha \in \mathbb{N}_0^N} f_\alpha z^\alpha$ , and for  $w \in \mathbb{B}_N$ , we have

$$(4.1) \quad \sqrt{1 - \|w\|^2} |f(w)| = |[f, e_w]| \leq \|f\|.$$

Let  $F = (f_{ij}) \in \mathbf{H}(\mathbb{B}_N)^{n \times m}$ , where the entries  $f_{ij} \in \mathbf{H}(\mathbb{B}_N)$  ( $i = 1, \dots, n$  and  $j = 1, \dots, m$ ), and let  $P$  denote a projection of rank  $r_0$ . Then:

$$\text{Tr } F(w)^* P B(w)^* B(w) P F(w) \leq \text{Tr } F(w)^* F(w) \quad (\text{since } B(w) \text{ is contractive inside the sphere})$$

$$= \sum_{i=1}^n \sum_{j=1}^m |f_{ij}(w)|^2.$$

Hence, using (4.1) for every  $f_{ij}$ , we obtain

$$(4.2) \quad (1 - \|w\|^2) (\text{Tr} [B(w)P F(w), B(w)P F(w)]) \leq \sum_{i=1}^n \sum_{j=1}^m \|f_{ij}\|^2 = \|F\|^2.$$

Let  $\epsilon > 0$ . In view of the power series expansion characterization (2.2) of the elements of the Drury–Arveson space, there exists a  $\mathbb{C}^{n \times m}$ -valued polynomial  $p$  in  $z_1, \dots, z_N$  such

that  $\|F - p\| \leq \epsilon$ . We have

$$\begin{aligned}
(1 - \|w\|^2) (\text{Tr} [B(w)PF(w), B(w)PF(w)]) & \\
& \leq (1 - \|w\|^2) (\text{Tr} [F(w), F(w)]) \\
& = (1 - \|w\|^2) \|(F - p)(w) + p(w)\|^2 \\
& \leq 2(1 - \|w\|^2) (\|(F - p)(w)\| + \|p(w)\|)^2 \\
& \leq 2(1 - \|w\|^2) \|(F - p)(w)\|^2 + 2(1 - \|w\|^2) \|p(w)\|^2 \\
& \leq 2\|F - p\|^2 + 2(1 - \|w\|^2) \|p(w)\|^2 \quad (\text{where we have used (4.1)}) \\
& \leq 2\epsilon^2 + 2(1 - \|w\|^2) \|p(w)\|^2.
\end{aligned}$$

Since  $(1 - \|w\|^2) \|p(w)\|^2$  tends to 0 as  $w$  approaches the unit sphere, the expression  $(1 - \|w\|^2) (\text{Tr} [B(w)PF(w), B(w)PF(w)])$  can be made arbitrary small, uniformly with respect to  $P$ , as  $w$  approaches the unit sphere. Thus,

$$(1 - \|w\|^2) (\text{Tr} [B(w)PF(w), B(w)PF(w)])$$

is uniformly bounded as  $w \in \mathbb{B}_N$  and  $P$  runs through the projections of rank  $r_0$ , and goes to 0 as  $w$  tends to the boundary. It has therefore a finite supremum, which is in fact a maximum and is in  $\mathbb{B}_N$  (and not on the boundary), as is seen by taking a subsequence tending to this supremum, and this ends the proof.  $\square$

Let us rewrite  $F(z)$  as

$$(4.3) \quad F(z) = P_0 F(w_0) e_{w_0}(z) \sqrt{1 - \|w_0\|^2} + F(z) - P_0 F(w_0) e_{w_0}(z) \sqrt{1 - \|w_0\|^2}.$$

We now show that (4.3) gives an orthogonal decomposition of  $F$ , which is the first step in the expansion of  $F$  that we are looking for (see (5.2) for a more precise way of writing the decomposition) and for the algorithm that will arise repeating this construction.

**Lemma 4.2.** *Let*

$$\begin{aligned}
H(z) &= F(z) - P_0 F(w_0) e_{w_0}(z) \sqrt{1 - \|w_0\|^2} \\
H_0(z) &= P_0 F(w_0) e_{w_0}(z) \sqrt{1 - \|w_0\|^2},
\end{aligned}$$

where  $w_0, P_0$  are as in Proposition 4.1. It holds that

$$(4.4) \quad P_0 H(w_0) = 0$$

and

$$[F, F] = [H_0, H_0] + [H, H].$$

*Proof.* First we have (4.4) since

$$P_0 H(w_0) = P_0 F(w_0) - P_0 F(w_0) e_{w_0}(w_0) \sqrt{1 - \|w_0\|^2} = 0.$$

Using (4.4) we have

$$[H, P_0 F(w_0) e_{w_0}(z) \sqrt{1 - \|w_0\|^2}] = F(w_0)^* P_0 H(w_0) (1 - \|w_0\|^2) = 0.$$

So,  $[H, H_0] = 0$  and

$$[F, F] = [H_0 + H, H_0 + H] = [H_0, H_0] + [H, H].$$

$\square$

## 5. THE ALGORITHM

To proceed and take care of the condition (4.4) (that is, in the scalar case, to divide by a Blaschke factor) we use a factor of the form (3.1). Then, we use Theorem 3.4 to find a  $\mathbb{C}^{n \times (n+r'_0(N-1))}$ -valued rational function  $B_{w_0, P_0}$  with  $r'_0 \leq r_0$  and such that

$$\text{ran } P_0 e_{w_0} = \mathbf{H}(\mathbb{B}_N)^{n \times m} \ominus B_{w_0, P_0}(\mathbf{H}(\mathbb{B}_N))^{(n+r'_0(N-1)) \times m},$$

and so

$$(5.1) \quad \mathbf{H}(\mathbb{B}_N)^{n \times m} = \left( \mathbf{H}(\mathbb{B}_N)^{n \times m} \ominus B_{w_0, P_0}(\mathbf{H}(\mathbb{B}_N))^{(n+r'_0(N-1)) \times m} \right) \oplus B_{w_0, P_0} \mathbf{H}(\mathbb{B}_N)^{(n+r'_0(N-1)) \times m}.$$

Let  $F \in (\mathbf{H}(\mathbb{B}_N))^{n \times m}$ . We choose  $w_0 \in \mathbb{B}_N$  and  $r_0 \in \{1, \dots, n\}$ . Using the maximum selection principle with  $B(z) = I_n$  we get a decomposition of the form (5.1). We rewrite (4.3) as

$$(5.2) \quad F(z) = P_0 F(w_0) e_{w_0}(z) \sqrt{1 - \|w_0\|^2} + B_{w_0, P_0}(z) F_1(z),$$

where  $F_1 \in (\mathbf{H}(\mathbb{B}_N))^{(n+r'_0(N-1)) \times m}$  (which, as  $F_1$  is uniquely defined when  $N > 1$ ). We now select  $w_1 \in \mathbb{B}_N$  and  $r_1 \in \{1, \dots, n+r'_0(N-1)\}$ , and apply the maximum selection principle to the pair  $(B_{w_0, P_0}(z), F_1(z))$ . We have then

$$(5.3) \quad F_1(z) = P_1 F_1(w_1) e_{w_1}(z) \sqrt{1 - \|w_1\|^2} + B_{w_1, P_1}(z) F_2(z),$$

where  $F_2 \in (\mathbf{H}(\mathbb{B}_N))^{(n+(r'_0+r'_1)(N-1)) \times m}$  (with  $r'_1 \leq r_1$ ) is not uniquely defined when  $N > 1$ . So

$$F(z) = P_0 F(w_0) e_{w_0}(z) \sqrt{1 - \|w_0\|^2} + B_{w_0, P_0}(z) P_1 F_1(w_1) e_{w_1}(z) \sqrt{1 - \|w_1\|^2} + B_{w_0, P_0}(z) B_{w_1, P_1}(z) F_2(z).$$

We iterate the procedure with the pair  $(B_{w_0, P_0}(z) B_{w_1, P_1}(z), F_2(z))$  and observe the appearance of the Blaschke product

$$\mathcal{B}_k(z) = B_{w_0, P_0} B_{w_1, P_1} B_{w_2, P_2} \cdots B_{w_{k-1}, P_{k-1}}, \text{ for } k \geq 1,$$

which will be  $\mathbb{C}^{n \times (1+s_k(N-1))}$ -valued for some  $s_k \leq \sum_{j=0}^{k-1} r_j$ . We set

$$(5.4) \quad M_k = F_k(w_k) \in \mathbb{C}^{s_k \times m},$$

and

$$\mathfrak{B}_k(z) = \begin{cases} \sqrt{1 - \|w_0\|^2} e_{w_0}(z) & \text{for } k = 0, \\ \sqrt{1 - \|w_k\|^2} e_{w_k}(z) B_{w_0, P_0}(z) B_{w_1, P_1}(z) B_{w_2, P_2}(z) \cdots B_{w_{k-1}, P_{k-1}}(z) & \text{for } k \geq 1. \end{cases}$$

Note that

$$(5.5) \quad \mathfrak{B}_k(w_k) = \mathcal{B}_k(w_k), \quad k \geq 1.$$

We have

$$(5.6) \quad F(z) = \sum_{k=0}^u \mathfrak{B}_k(z) M_k + \mathcal{B}_{u+1}(z) F_{u+1}(z).$$

Moreover,

$$(5.7) \quad \langle \mathfrak{B}_k M_k, \mathfrak{B}_\ell M_\ell \rangle_{\mathbf{H}(\mathbb{B}_N)} = 0 \quad \text{for } k \neq \ell$$



and we have by the orthogonality of the decomposition that

$$(5.8) \quad \|F\|_{\mathbf{H}(\mathbb{B}_N)}^2 = \sum_{k=0}^u \|\mathfrak{B}_k M_k\|_{\mathbf{H}(\mathbb{B}_N)}^2 + \|\mathcal{B}_{u+1}(z)F_{u+1}\|_{\mathbf{H}(\mathbb{B}_N)}^2.$$

This recursive procedure gives, at the  $k$ -th step, the best approximation. However we have to ensure that when  $k$  tends to infinity the algorithm converges. This is guaranteed by virtue of the next result.

**Theorem 5.1.** *Suppose that in (5.6) at each step one selects  $w_k$  and  $P_k$  according to the maximum selection principle applied to  $(\mathcal{B}_{w_k}(z), F_k(z))$ . Then the algorithm converges, meaning that*

$$F(z) = \sum_{k=0}^{\infty} \mathfrak{B}_k(z) M_k$$

in the norm of the Drury-Arveson space.

*Proof.* We follow the arguments of [15] and [4]. We set

$$(5.9) \quad R_u(z) = F(z) - \sum_{k=0}^u \mathfrak{B}_k(z) M_k = \mathcal{B}_{w_{u+1}}(z) F_{u+1}(z)$$

(where  $F_{u+1}$  is not uniquely defined when  $N > 1$ ) and

$$S_u(z) = \sum_{k=u+1}^{\infty} \mathfrak{B}_k(z) M_k.$$

In view of (5.7)-(5.8) the sum  $\sum_{k=0}^{\infty} \mathfrak{B}_k(z) M_k$  converges in the Drury-Arveson space. Let  $G$  be its limit, and assume that  $G \neq F$ . Thus there exists  $w \in \mathbb{B}_N$  such that  $G(w) \neq F(w)$ . We now proceed in a number of steps to obtain a contradiction.

STEP 1: *There exists  $u_0 \in \mathbb{N}$  such that for  $u \geq u_0$*

$$(5.10) \quad \sqrt{1 - \|w\|^2} \cdot \|R_u(w)\| > \sup_{\substack{c \in \mathbb{C}^n, \|c\|=1 \\ d \in \mathbb{C}^m, \|d\|=1}} \frac{|\langle (F - G)d, ce_w \rangle_{(\mathbf{H}(\mathbb{B}_N))^n}|}{2}.$$

Indeed,  $S_u$  tends to 0 in norm in  $(\mathbf{H}(\mathbb{B}_N))^{n \times m}$ . Since in a reproducing kernel Hilbert space convergence in norm implies pointwise convergence, we have  $\lim_{u \rightarrow \infty} S_u(w) = 0_{n \times m}$  in the norm of  $\mathbb{C}^{n \times m}$ , and there exists  $u_0 \in \mathbb{N}$  such that

$$u \geq u_0 \implies \|S_u(w)\| < \frac{\|F(w) - G(w)\|}{2}.$$

Thus

$$\|R_u(w)\| + \frac{\|F(w) - G(w)\|}{2} > \|R_u(w)\| + \|S_u(w)\| \geq \|F(w) - G(w)\|,$$

and so

$$\|R_u(w)\| > \frac{\|F(w) - G(w)\|}{2},$$

which can be rewritten as (5.10).

STEP 2: *It holds that*

$$(5.11) \quad \lim_{k \rightarrow \infty} (1 - \|w_k\|^2) \|\mathfrak{B}_k(w_k) M_k\|^2 = 0$$

Indeed, from the convergence of  $\sum_{k=0}^{\infty} \mathfrak{B}_k M_k$  we have

$$\lim_{k \rightarrow \infty} \|\mathfrak{B}_k M_k\|_{(\mathbf{H}(\mathbb{B}_N))^{n \times m}} = 0.$$

Thus, with  $c \in \mathbb{C}^m$  and  $d \in \mathbb{C}^n$ , we have:

$$\begin{aligned} |\langle \mathfrak{B}_k(w_k) M_k c, d \rangle_{(\mathbf{H}(\mathbb{B}_N))^n}| &= |\langle \mathfrak{B}_k M_k c, \frac{d}{1 - \langle \cdot, w_k \rangle} \rangle_{(\mathbf{H}(\mathbb{B}_N))^n}| \\ &\leq \|\mathfrak{B}_k M_k c\|_{(\mathbf{H}(\mathbb{B}_N))^n} \cdot \frac{\|d\|}{\sqrt{1 - \|w_k\|^2}} \\ &\leq \|\mathfrak{B}_k M_k\|_{(\mathbf{H}(\mathbb{B}_N))^{n \times m}} \cdot \|c\| \cdot \frac{\|d\|}{\sqrt{1 - \|w_k\|^2}}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality. So, after taking supremum on  $c$  and  $d$ ,

$$\|\sqrt{1 - \|w_k\|^2} \mathfrak{B}_k(w_k) M_k\| \leq \|\mathfrak{B}_k M_k\|_{(\mathbf{H}(\mathbb{B}_N))^{n \times m}} \longrightarrow 0 \text{ as } n \rightarrow \infty,$$

and so (5.11) holds in view of (5.5).

STEP 3: *We conclude the proof.*

Let  $u \geq u_0$ , where  $u_0$  is as in Step 1. Since  $R_u(z) = \mathcal{B}_{w_{u+1}}(z) F_{u+1}(z)$  and since  $w$  is such that  $F(w) \neq G(w)$  we have

$$(5.12) \quad \sqrt{1 - \|w\|^2} \cdot \|\mathcal{B}_{w_{u+1}}(w) F_{u+1}(w)\| > \sup_{\substack{c \in \mathbb{C}^n, \|c\|=1 \\ d \in \mathbb{C}^m, \|d\|=1}} \frac{|\langle (F - G)d, ce_w \rangle_{(\mathbf{H}(\mathbb{B}_N))^n}|}{2}.$$

By definition of  $w_{u+1}$  we have

$$\sqrt{1 - \|w_{u+1}\|^2} \cdot \|\mathcal{B}_{w_{u+1}}(w_{u+1}) F_{u+1}(w_{u+1})\| < \sup_{\substack{c \in \mathbb{C}^n, \|c\|=1 \\ d \in \mathbb{C}^m, \|d\|=1}} \frac{|\langle (F - G)d, ce_w \rangle_{(\mathbf{H}(\mathbb{B}_N))^n}|}{2},$$

and using (5.5) we contradict (5.11). □

## 6. INFINITE BLASCHKE PRODUCTS

In the previous sections appeared the counterpart of finite Blaschke products in the setting of the ball. We now consider the case of infinite products.

Let  $a \in \mathbb{B}_N$ , and let  $b_a(z)$  be a  $\mathbb{C}^{1 \times N}$ -valued Blaschke factor. We use the formula

$$(6.1) \quad b_a(z) = \frac{a - \frac{za^*}{aa^*}a - \sqrt{1 - aa^*} \left( z - \frac{za^*}{aa^*}a \right)}{1 - za^*},$$

from [16, (2), p. 25] rather than the formula in (2.5). See [7, Lemma 4.2, p. 13] for the equality between the two expressions.

We first prove a technical lemma useful in the proof of the convergence of an infinite Blaschke product.

**Lemma 6.1.** *Let  $\alpha = \frac{-a}{\sqrt{aa^*}} \in \partial\mathbb{B}_N$ . Then,*

(6.2)

$$b_a(z) - b_a(\alpha) = \frac{(z - \alpha) \left( a^*a \left( \frac{1 - \sqrt{1 - aa^*}}{aa^*} \right) - I_N \right) + z(\alpha a^*) - \alpha(za^*)}{(1 - za^*)(1 + \sqrt{aa^*})} \cdot \sqrt{1 - aa^*}$$

and

$$(6.3) \quad \|b_a(z) - b_a(\alpha)\| \leq \frac{4\sqrt{1 - aa^*}}{1 - \|z\|}.$$

*Proof.* We write  $b_a(z) - b_a(\alpha) = \frac{\Delta}{(1 - za^*)(1 - \alpha a^*)}$ , where the numerator

$$\begin{aligned} \Delta = & \left( a - \frac{za^*}{aa^*}a - \sqrt{1 - aa^*} \left( z - \frac{za^*}{aa^*}a \right) \right) (1 - \alpha a^*) - \\ & - \left( a - \frac{\alpha a^*}{aa^*}a - \sqrt{1 - aa^*} \left( \alpha - \frac{\alpha a^*}{aa^*}a \right) \right) (1 - za^*) \end{aligned}$$

has 16 terms. Out of there,  $a$  and  $-a$  cancel each other, and

$$\frac{za^*}{aa^*}a(\alpha a^*) = \frac{\alpha a^*}{aa^*}a(za^*)$$

and

$$\sqrt{1 - aa^*} \frac{za^*}{aa^*}a(\alpha a^*) = \sqrt{1 - aa^*} \frac{\alpha a^*}{aa^*}a(za^*).$$

We are thus left with 10 terms, which can be rewritten as:

$$\Delta = (z - \alpha) \left( \left( -\frac{a^*a}{aa^*} - \sqrt{1 - aa^*}I_N + \sqrt{1 - aa^*} \frac{a^*a}{aa^*} + a^*a \right) + \sqrt{1 - aa^*} (z(\alpha a^*) - \alpha(za^*)) \right).$$

Note that  $z(\alpha a^*) - \alpha(za^*)$  does not vanish when  $N > 1$ . Therefore

(6.4)

$$\begin{aligned} b_a(z) - b_a(\alpha) = & \\ = & \frac{(z - \alpha) \left( \left( -\frac{a^*a}{aa^*} - \sqrt{1 - aa^*}I_N + \sqrt{1 - aa^*} \frac{a^*a}{aa^*} + a^*a \right) + \sqrt{1 - aa^*} (z(\alpha a^*) - \alpha(za^*)) \right)}{(1 - za^*)(1 + \sqrt{aa^*})}. \end{aligned}$$

□

**Remark 6.2.** We note that

$$\left\| a^*a \left( \frac{1 - \sqrt{1 - aa^*}}{aa^*} \right) - I_N \right\| = \sqrt{1 - aa^*},$$

as can be seen by computing the eigenvalues of the matrix in the left hand side.

We now consider a term of the form (3.1) and write (where  $\alpha = -\frac{a}{\sqrt{aa^*}}$  and  $W$  is a unitary matrix to be determined)

$$(6.5) \quad \begin{aligned} B_a(z) &= B(z)W \\ &= \left( U \begin{pmatrix} b_a(\alpha) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix} + U \begin{pmatrix} (b_a(z) - b_a(\alpha)) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix} \right) W, \end{aligned}$$

where we do not stress the dependence on the matrices  $U$  and  $W$ . Since  $b_a(\alpha)$  is a unit vector, the matrix

$$U \begin{pmatrix} b_a(\alpha) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix}$$

is coisometric, and we can complete the columns of its adjoint to a unitary matrix  $W$ . Then we have

$$(6.6) \quad U \begin{pmatrix} b_a(\alpha) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix} W = \begin{pmatrix} I & 0 \end{pmatrix}$$

and show that the corresponding infinite product will converge when  $\sum_{n=1}^{\infty} \sqrt{1 - a_n a_n^*}$  converges.

In Theorem 6.3 below we imbed  $\mathbb{C}^m$  inside  $\ell_2$  via the formula:

$$(6.7) \quad i_m(z_1, \dots, z_m) = (z_1, \dots, z_m, 0, 0, \dots).$$

We also need some notation and introduce the matrices

$$E_k = \begin{pmatrix} 1 & 0_{1 \times k(N-1)} \end{pmatrix} \quad (= 1 \text{ when } N = 1),$$

$$F_k = \begin{pmatrix} I_{1+(k-1)(N-1)} & 0_{(1+(k-1)(N-1)) \times (N-1)} \end{pmatrix} \in \mathbb{C}^{(1+(k-1)(N-1)) \times (N+(k-1)(N-1))},$$

and note that  $E_1 = F_1$  and

$$(6.8) \quad E_k = E_1 F_2 \cdots F_k \quad \text{and} \quad E_{k+1} = E_k F_{k+1}.$$

We also note that multiplication by  $F_k$  on the right imbeds  $\mathbb{C}^{1+(k-1)(N-1)}$  into  $\mathbb{C}^{1+k(N-1)}$ . It will be useful to use the notation

$$(6.9) \quad F_{m_1}^{m_2} = \prod_{k=m_1+1}^{\widehat{m_2}} E_k.$$

**Theorem 6.3.** *The infinite product  $b_{w_0}(z)B_{w_1}(z)B_{w_2}(z) \cdots B_{w_{k-1}}(z) \cdots$  where the factors are normalized as in (6.6) converges pointwise for  $z \in \mathbb{B}_N$  to a non-identically vanishing  $\ell_2$ -valued function analytic in  $\mathbb{B}_N$  if*

$$(6.10) \quad \sum_{k=0}^{\infty} \sqrt{1 - a_k a_k^*} < \infty.$$

*Proof.* The idea is to follow the proof for the scalar case appearing in sources such as [11, 13] and reproduced in [3, pp. 104-105]. We consider the product

$$\prod_{k=1}^{\widehat{m}} (F_k + A_k(z))$$

with

$$A_k(z) = U_k \begin{pmatrix} b_{a_k}(z) - b_{a_k}(\alpha) & 0_{1 \times (k-1)(N-1)} \\ 0_{(k-1)(N-1) \times N} & I_{(k-1)(N-1)} \end{pmatrix} W_k \in \mathbb{C}^{(1+(k-1)(N-1)) \times (N+(k-1)(N-1))},$$

and note that, in view of (6.3),

$$(6.11) \quad \|A_k(z)\| \leq \frac{4\sqrt{1 - a_k a_k^*}}{1 - \|z\|}.$$

Following the classical proof we now prove the convergence in a number of steps and use [3, pp. 104-105] as a source.

Note that, to ease the notation, in Steps 1-3 we do not stress the dependence of  $A_k$  on the variable  $z$ .

STEP 1: *It holds that*

$$(6.12) \quad \left\| \prod_{k=1}^{\widehat{m}} (F_k + A_k) - E_m \right\| \leq \prod_{k=1}^m (1 + \|A_k\|) - 1, \quad m \in \mathbb{N}.$$

We proceed by induction, the case  $m = 1$  being trivial since  $E_1 = F_1$ . We have

$$\begin{aligned} \left\| \prod_{k=1}^{\widehat{m+1}} (F_k + A_k) - E_{m+1} \right\| &= \left\| \left( \prod_{k=1}^m (F_k + A_k) \right) (F_{m+1} + A_{m+1}) - E_{m+1} \right\| \\ &= \left\| \left( \prod_{k=1}^{\widehat{m}} (F_k + A_k) \right) (F_{m+1} + A_{m+1}) - E_m F_{m+1} \right\| \\ &\leq \left\| \left( \left( \prod_{k=1}^{\widehat{m}} (F_k + A_k) \right) - E_m \right) F_{m+1} \right\| + \\ &\quad + \|A_{m+1}\| \left\| \prod_{k=1}^m (1 + \|A_k\|) \right\| \\ &\leq \left( \left( \prod_{k=1}^n (1 + \|A_k\|) \right) - 1 \right) + \|A_{m+1}\| \left( \prod_{k=1}^m (1 + \|A_k\|) \right) \\ &= \left( \prod_{k=1}^{m+1} (1 + \|A_k\|) \right) - 1, \end{aligned}$$

where we have used the induction hypothesis to go from the third to the fourth line.

Replacing  $A_k$  by  $A_{k+m_1}$  we have for  $m_2 > m_1$ :

$$(6.13) \quad \left\| \left( \prod_{k=m_1+1}^{\widehat{m_2}} (E_k + A_k) \right) - \prod_{k=m_1+1}^{\widehat{m_2}} E_{m_2} \right\| \leq \left( \prod_{k=m_1+1}^{m_2} (1 + \|A_k\|) \right) - 1.$$

STEP 2: *Let  $Z_m = \prod_{k=1}^{\widehat{m}} (F_k + A_k)$ . Then,*

$$\|Z_m\| \leq e^{\sum_{k=1}^m \|A_k\|} < \infty$$

Indeed,

$$\begin{aligned}
\|Z_m\| &\leq \prod_{k=1}^m \|F_k + A_k\| \\
&\leq \prod_{k=1}^m (1 + \|A_k\|) \\
&\leq \prod_{k=1}^m e^{\|A_k\|} \leq e^{\sum_{k=1}^{\infty} \|A_k\|} < \infty,
\end{aligned}$$

in view of (6.10) and (6.11).

STEP 3: Let  $i_m$  be defined by (6.7). Then,  $(i_m(Z_m))_{m \in \mathbb{N}}$  is a Cauchy sequence in  $\ell_2$ .

For  $m_2 > m_1$  and using (6.13), we have

$$\begin{aligned}
\|i_{m_2}(Z_{m_2}) - i_{m_1}(Z_{m_1})\|_{\ell_2} &= \|Z_{m_2} - i_{m_2} \cdots i_{m_1+1}(Z_{m_1})\|_{\mathbb{C}^{1 \times (1+(m_2+1)(N-1))}} \\
&= \left( \prod_{k=1}^{\widehat{m_1}} (F_k + A_k) \right) \cdot \left( \prod_{k=m_1+1}^{\widehat{m_2}} (F_k + A_k) - F_{m_1+1}^{m_2} \right) \\
&\leq \left( \prod_{k=1}^{m_1} (1 + \|A_k\|) \right) \cdot \left\| \prod_{k=m_1+1}^{\widehat{m_2}} (F_k + A_k) - F_{m_1+1}^{m_2} \right\| \\
(6.14) \quad &\leq e^K \left\{ \left( \prod_{k=m_1+1}^{m_2} (1 + \|A_k\|) \right) - 1 \right\} \\
&\leq e^K \left\{ \left( \prod_{k=m_1+1}^{m_2} e^{\|A_k\|} \right) - 1 \right\} \\
&\leq \left( \sum_{k=m_1+1}^{m_2} \|A_k\| \right) e^{2K},
\end{aligned}$$

with  $K = \sum_{k=1}^{\infty} \|A_k\|$  (which is finite, thanks to (6.10) and (6.11)), and using inequality

$$e^x \leq 1 + xe^x, \quad x \geq 0,$$

with  $x = \sum_{k=m_1+1}^{m_2} \|A_k\|$ .

STEP 4: The  $\ell_2$ -valued function  $Z(z) = \lim_{m \rightarrow \infty} Z_m(z)$  does not vanish identically in  $\mathbb{B}_N$ .

We first assume that  $\sum_{k=1}^{\infty} \|A_k(z)\| < \frac{1}{2}$  and prove by induction that

$$(6.15) \quad \|Z_m(z)\| \geq 1 - \sum_{k=1}^m \|A_k(z)\|.$$

The claim  $Z \neq 0$  will then follow by letting  $m \rightarrow \infty$ . For  $m = 1$  the claim is trivial. Assume that (6.15) holds for  $m$ . We then have:

$$\begin{aligned}
\|Z_{m+1}(z)\| &= \|Z_m(z)(F_{m+1} + A_{m+1}(z))\| \\
&\geq \|Z_m(z)F_{m+1}\| - \|Z_m(z)A_{m+1}(z)\| \quad (\text{since } \|Z_m(z)F_{m+1}\| = \|Z_m(z)\|) \\
&\geq \|Z_m(z)\| - \|Z_m(z)\| \|A_{m+1}(z)\| \quad (\text{since } \|Z_m(z)A_{m+1}(z)\| \leq \|Z_m(z)\| \|A_{m+1}(z)\|) \\
&= \|Z_m(z)\| \cdot (1 - \|A_{m+1}(z)\|) \\
&\geq (1 - \sum_{k=1}^m \|A_k(z)\|)(1 - \|A_{m+1}(z)\|) \\
&\geq (1 - \sum_{k=1}^{m+1} \|A_k(z)\|).
\end{aligned}$$

Let  $M \in \mathbb{N}$  (depending on  $z$ ) be such that  $\sum_{k=M}^{\infty} \|A_k(z)\| < \frac{1}{2}$ . Then the same inequality holds in an open neighborhood  $V$  of  $z$  in view of (6.11), and so the same  $M$  can be taken for  $z \in V$ . Let

$$Z_{M-1}(z) = \prod_{u=1}^{\infty} (F_u + A_u(z)) \in \mathbb{C}^{1 \times (1+(M-2)(N-1))},$$

where

$$\widetilde{Z}_M(z) = \prod_{u=M}^{\infty} (F_u + A_u(z)).$$

We can patch together all the  $Z_{M-1}(z)\widetilde{Z}_M(z)$  to a common function defined in  $\mathbb{B}_N$ . Assume that  $Z_{M-1}(z)\widetilde{Z}_M(z) \equiv 0$  in one of the neighborhoods  $V$ . Then the infinite product vanishes identically in  $\mathbb{B}_N$ . Letting  $z$  go to the boundary we get a contradiction since  $Z_{M-1}(z)\widetilde{Z}_M(z)$  takes coisometric values on  $\partial\mathbb{B}_N$ .

STEP 5: Using (6.12) and (6.14), we obtain the bound:

$$(6.16) \quad \left\| \prod_{k=1}^{\infty} (F_k + A_k(z)) - Z \right\| \leq e^{2K} \left( \sum_{k=m+1}^{\infty} \|A_k(z)\| \right).$$

□

It is worthwhile to note that the above theorem allows to further extending the results of [7] to the case of an infinite number of points.

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